

Emergent Classicality via Commuting Position and Momentum Operators

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Abstract

Any account of the emergence of classicality from quantum theory must address the fact that the quantum operators representing positions and momenta do not commute, whereas their classical counterparts suffer no such restrictions. To address this, we revive an old idea of von Neumann, and seek a pair of commuting operators X, P which are, in a specific sense, “close” to the canonical non-commuting position and momentum operators, x, p . The construction of such operators is related to the problem of finding complete sets of orthonormal phase space localized states, a problem severely limited by the Balian-Low theorem. Here these limitations are avoided by restricting attention to situations in which the density matrix is reasonably decohered (i.e., spread out in phase space).

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I. INTRODUCTION

A. Opening Remarks

The canonical commutation relation between position and momentum operators,

$$[\hat{x}, \hat{p}] = i\hbar \tag{1.1}$$

lies at the heart of all quantum phenomena. Physically, this relation corresponds to the fact that measurements of position and momentum depend on the order of the measurements. It is this relation, in essence, that is responsible for the key differences between classical and quantum mechanics, since in classical mechanics, measurements of position and momentum can be made in such a way that the order makes no difference. It follows that any account of the emergence of classical behaviour from quantum theory must reconcile these two very different aspects of classical and quantum theory. A typical point of view is that classical mechanics emerges only at a very coarse-grained level and for sufficiently coarse-grained samplings of position and momentum, their non-commutativity makes little difference [1].

Still, one wonders whether there is a deeper or more precise way of reconciling the non-commuting quantum operators with their commuting classical counterparts. Indeed, this clearly troubled the founders of quantum theory, since von Neumann addressed the issue in his 1932 book, *Mathematical Foundations of Quantum Mechanics* [2]. He noted that when we make observations of a macroscopic system, we are in fact able to make observations of position and momentum simultaneously (although imprecisely, of course). This suggested to him that the measurements we make do not in fact correspond directly to the usual operators \hat{x} and \hat{p} , but to some other operators, \hat{X} and \hat{P} , say, which commute

$$[\hat{X}, \hat{P}] = 0 \tag{1.2}$$

and which must in some sense be “close” to the original operators \hat{x} , \hat{p} . Such a pair of operators could be particularly useful in bringing a degree of precision to discussions of emergent classicality. The aim of this paper is to discuss the construction and utility of such operators.

B. The Imprecision of Classical Physics

To what degree do macroscopic measurements fix the position and momentum operators in quantum mechanics? When we make macroscopic measurements of, say, a particle, there will be imprecisions Δx , Δp in the specifications of position and momentum. These imprecisions may be “small” to classical eyes but they will typically be very large compared to the quantum scale. It will be useful for what follows to get a quantitative idea of this. Suppose that the position is measured to a precision of $10^{-8}m$ and the velocity to within $10^{-8}m/s$ (both extremely precise specifications, by macroscopic standards). For a mass of, say, $10^{-6}kg$, we then have,

$$\frac{\Delta p \Delta x}{\hbar} \sim 10^{12} \quad (1.3)$$

This means that when we specify the phase space location of a classical system in a way that is very precise to classical eyes, at the quantum scale it could be in any one of about 10^{12} phase space cells. There is therefore a considerable amount of freedom at the quantum level to redefine the position and momentum operators without making any noticeable difference to macroscopic observations. Perhaps within this freedom there is the possibility to find position and momentum operators which commute.

C. Motivations

What are the motivations for constructing commuting operators which are close to the position and momentum operators? An important indicator of emergent classicality is approximate diagonality of the density operator ρ (often referred to as decoherence) [3]. For example, for a point particle interacting with a thermal environment, it may be shown that after a short time scale, the density operator approaches the form

$$\rho = \int dp dq f(p, q) |\psi_{pq}\rangle \langle \psi_{pq}| \quad (1.4)$$

where $f(p, q)$ is a non-negative function and $|\psi_{pq}\rangle$ are a set of phase space localized states (such as generalized coherent states) [4]. This indicates that it is approximately diagonal in both position or momentum. (It cannot be exactly diagonal in both, since they do not commute.) This means that there is negligible interference between different values of position and momenta and, loosely speaking, they may then be treated as if they are

classical. The statement is imprecise, however, since there is still *some* interference, so the variables are only imprecisely defined.

One would like to be able to make a more exact statement about diagonality of ρ and this is possible in terms of the commuting operators X, P . For example, using the eigenstates $|\psi_{nm}\rangle$ of X, P one could construct a density operator of the form

$$\rho' = \sum_{nm} f_{nm} |\psi_{nm}\rangle \langle \psi_{nm}| \quad (1.5)$$

which is exactly diagonal in X and P , indicating there is exactly *zero* interference between different values of these quantities. One can then choose the coefficients f_{nm} to make ρ' as close as possible to ρ . The question of the degree of approximate diagonality is therefore shifted to the question of the closeness of the density operator to a pseudo-classical density operator ρ' .

A more comprehensive approach to emergent classicality is the decoherent histories approach [5, 6, 7, 8, 9, 10]. There, the central object of interest is the decoherence functional,

$$D(\underline{\alpha}, \underline{\alpha}') = \text{Tr} (P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1) \rho P_{\alpha'_1}(t_1) \cdots P_{\alpha'_n}(t_n)) \quad (1.6)$$

The histories are characterized by the initial state ρ and by the strings of projection operators $P_{\alpha}(t)$ (in the Heisenberg picture) at times t_1 to t_n (and $\underline{\alpha}$ denotes the string of alternatives $\alpha_1 \cdots \alpha_n$). Intuitively, the decoherence functional is a measure of the interference between pairs of histories $\underline{\alpha}, \underline{\alpha}'$. When

$$\text{Re} D(\underline{\alpha}, \underline{\alpha}') = 0 \quad (1.7)$$

for $\underline{\alpha} \neq \underline{\alpha}'$, we say that the histories are consistent and probabilities $p(\underline{\alpha}) = D(\underline{\alpha}, \underline{\alpha})$ obeying the usual probability sum rules may be assigned to them. (Typically, the physical mechanisms producing consistency actually cause the stronger condition $D(\underline{\alpha}, \underline{\alpha}') = 0$ for $\underline{\alpha} \neq \underline{\alpha}'$ to be satisfied, which is referred to as decoherence.) One can then ask whether these probabilities are strongly peaked about trajectories obeying classical equations of motion, and if they are, we can say that the system is emergently classical.

For histories in which the projections $P_{\alpha_k}(t_k)$ are onto positions at different times of a point particle interacting with a thermal environment, the decoherence functional, like the density operator, is *approximately* diagonal [6, 10]. The approximation is typically exceptionally good, but still only approximate. Again one wonders whether more exact statements can be made. Indeed, it has been conjectured that approximately consistent

histories can be in some sense distorted into exactly consistent ones [11]. There are a number of ways in which a set of histories could be distorted: one can change the initial state, the times of the projections, the widths of the projections or the operators whose spectrum is projected onto. The possible existence of the commuting variables \hat{P} , \hat{X} suggests a particular way of distorting the histories so as to make them exactly decoherent.

Each position at time t is, in the Heisenberg picture, a function of the canonical pair, \hat{x}, \hat{p} , so $\hat{x}_t = f_t(\hat{x}, \hat{p})$. The projections onto \hat{x}_t at different times do not commute, since $[\hat{x}_t, \hat{x}_{t'}] \neq 0$ in general. The decoherence functional is therefore not diagonal in general, but can be approximately diagonal if the system is coupled to a thermal environment.

Now suppose we consider projections onto the variables $\hat{X}_t = f_t(\hat{X}, \hat{P})$ at different times. Under reasonable dynamics, \hat{X}_t will be close to \hat{x}_t as long as \hat{x}, \hat{p} are close to \hat{X}, \hat{P} . The operators \hat{X}_t do commute at different times so all the projections commute and, as is easy to see, the decoherence functional will be exactly diagonal. So exact decoherence is achieved by shifting the operators \hat{x}, \hat{p} to the commuting pair \hat{X}, \hat{P} . Furthermore, it is known that the probabilities for histories of positions \hat{x}_t are typically peaked about classical evolution. The probabilities for the commuting variables \hat{X}_t will therefore have the same property if the \hat{X}_t are close to \hat{x}_t . The question of approximate decoherence and emergent classicality is therefore shifted to the question of the closeness of the old and new operators.

These then, at least in outline, are the reasons why a commuting pair of position and momentum-like operators may be useful for discussing emergent classicality. We turn now to the construction of such operators.

D. Von Neumann's Construction

Von Neumann outlined a prescription whereby the commuting operators \hat{X} and \hat{P} may be constructed [2]. This involved first taking a discrete subset $|m, n\rangle$ of the coherent states $|p, q\rangle$, with one state per cell of size $2\pi\hbar$ (a von Neumann lattice). He alleged that these states are complete (this was later proved [12, 13, 14]). He then stated that they may be orthogonalized using the Schmidt process to produce an orthonormal set $|\psi_{nm}\rangle$. From these, he constructed position and momentum-like operators

$$\hat{X} = \sum_{nm} na |\psi_{nm}\rangle \langle \psi_{nm}|, \quad \hat{P} = \sum_{nm} m \frac{2\pi\hbar}{a} |\psi_{nm}\rangle \langle \psi_{nm}| \quad (1.8)$$

(where a is a constant with the dimension of length). These operators clearly commute. He argued that these new operators are indeed “close” to the old ones, in the sense that

$$\langle \psi_{nm} | (\hat{x} - \hat{X})^2 | \psi_{nm} \rangle \langle \psi_{nm} | (\hat{p} - \hat{P})^2 | \psi_{nm} \rangle \leq K^2 \hbar^2 \quad (1.9)$$

Von Neumann’s calculations imply that the constant K is about 1,800 (but he thought more detailed calculations could give a smaller value).

However, von Neumann’s prescription is at best a sketch of how this works and he certainly did not give full details (such as the explicit form of the $|\psi_{nm}\rangle$). Furthermore, we now know a lot more about phase space localized states than was known in 1932, and, as will be described below, there are obstructions to constructing such states. These obstructions do not necessarily apply to what von Neumann did, but nevertheless, it is still interesting to revisit his ideas from a more modern perspective.

E. A General Approach and the Balian-Low Theorem

We start with a more general statement of the problem. We consider a set of states of the form

$$|\psi_{nm}\rangle = U_{nm}|\psi\rangle \quad (1.10)$$

where $|\psi\rangle$ is a fiducial state and U_{nm} is the unitary shift operator,

$$U_{nm} = \exp\left(\frac{i}{\hbar}na\hat{p} - \frac{i}{\hbar}mb\hat{x}\right) \quad (1.11)$$

A particularly interesting case is that of a von Neumann lattice, in which case $b = 2\pi\hbar/a$ and the translations in the p and x directions then commute, and we have

$$U_{nm} = (-1)^{mn} \exp\left(\frac{i}{\hbar}na\hat{p}\right) \exp\left(-i\frac{2\pi m}{a}\hat{x}\right) \quad (1.12)$$

There is then one state per cell of size $2\pi\hbar$, as in von Neumann’s case. It is of interest to find a fiducial state $|\psi\rangle$ in Eq.(1.10) such that the states are complete and orthonormal, that is,

$$\sum_{nm} |\psi_{nm}\rangle \langle \psi_{nm}| = 1 \quad (1.13)$$

$$\langle \psi_{nm} | \psi_{n'm'} \rangle = \delta_{nn'} \delta_{mm'} \quad (1.14)$$

Fiducial states leading to states satisfying these properties are easily found and we will exhibit a set below. There is, however, a crucial difficulty. According to the theorem of

Balian and Low [15, 16, 17], if the three properties (1.10), (1.13) and (1.14) are satisfied, then the fiducial state $|\psi\rangle$ has either $(\Delta x)^2$ or $(\Delta p)^2$ infinite, so is not phase space localized. If we used such states to construct commuting operators \hat{X}, \hat{P} as von Neumann did, then at least one of the averages in Eq.(1.9) would diverge, and there would be no sense in which the new operators are close to the old. (Note that von Neumann claims to have used the Schmidt procedure to construct his orthonormal set, which one would not expect to produce states satisfying Eq.(1.10), so his construction does not necessarily fall foul of the Balian-Low theorem).

The problem of constructing orthonormal phase space localized states is one of great interest in a number of fields so some effort has been expended in finding ways around the Balian-Low theorem. Zak has proved some interesting results in this area. In Ref [18], he showed that the coherent states restricted to a von Neumann lattice $|m, n\rangle$ obey a sort of orthogonality relation if the usual inner product $\langle m, n | m', n' \rangle$ is averaged over a single phase space cell. It is not yet clear if this result can be used to produce commuting position and momentum operators. He has also considered complete orthonormal sets of states which are localized in position, but double-peaked in momentum (so localized in p^2 , but not in p). From these one can construct commuting operators \hat{X} and \hat{P}^2 , which are “close” to \hat{x} and \hat{p}^2 [19]. This is tantalizing close to the goal of this paper, but not quite there (and also suggests that the $p \rightarrow -p$ transformation plays a crucial role in the Balian-Low theorem). Many of these and similar results are proved using the so-called kq representation, a technique which is particularly well-adapted to these problems [20].

To avoid the Balian-Low theorem, one has to drop one of the three requirements (1.10), (1.13) and (1.14) in order to get phase space localization. For the purposes of this paper, which is to construct useful commuting position and momentum operators, the orthogonality Eq.(1.14) is essential. We will therefore explore the possibility of dropping the other two requirements. Eq.(1.10), the requirement that the states be obtained by translation of a single fiducial state is mainly for practical convenience, so there is no harm in relaxing this as long as the resulting states are not unmanageable.

More significantly, we will relax the requirement of completeness, Eq.(1.13). The motive behind this is as follows. We are interested in using commuting operators to discuss emergent classicality. In practice, this means that we are only concerned with the physical situation in which the density matrix of the system has undergone a degree of decoherence. This

means that it is approximately diagonal in both position and momentum, or equivalently, its Wigner function is reasonably spread out in phase space. The density matrix is therefore insensitive to the fine structure of phase space and it seems reasonable to suppose that the physics could be well-described by a less than complete set of states, if they are carefully chosen. Of course, this is a quantitative matter that needs to be checked in detail and we will do this.

For the purposes of constructing commuting position and momentum operators, we will actually use a construction slightly more general than the one indicated by von Neumann in Eq.(1.8). In particular, we will look for commuting operators \hat{X} , \hat{P} of the form

$$\hat{X} = \sum_{nm} X_n E_{nm} \quad (1.15)$$

$$\hat{P} = \sum_{nm} P_m E_{nm} \quad (1.16)$$

Here, X_n and P_m are c -numbers and E_{nm} are projection operators localized onto a region of phase space labelled by n, m which will consist of *more than one* $2\pi\hbar$ -sized cell. They are exclusive

$$E_{nm} E_{n'm'} = E_{nm} \delta_{nn'} \delta_{mm'} \quad (1.17)$$

and exhaustive

$$\sum_{nm} E_{nm} = 1 \quad (1.18)$$

We will also insist that they are obtained from unitary shifts of a single projector E ,

$$E_{nm} = U_{nm} E (U_{nm})^\dagger \quad (1.19)$$

Here, U_{nm} shifts from one cell to the next, so is of the form Eq.(1.11), with a, b chosen so that the translation in the position and momentum directions commute, but $ab > 2\pi\hbar$. These three conditions are the natural generalization for projection operators of the requirements (1.10), (1.13) and (1.14), and the original case is obtained with the choice

$$E_{nm} = |\psi_{nm}\rangle \langle \psi_{nm}| \quad (1.20)$$

The quantity $\text{Tr} E$ is a measure of the number of $2\pi\hbar$ -sized cells projected onto, so clearly $\text{Tr} E = 1$ in the pure state case, but $\text{Tr} E \gg 1$ in our case, as we will see.

One might have thought that this more general construction could avoid the Balian-Low theorem, since the restrictions on E_{nm} are in fact weaker than the restrictions (1.10), (1.13)

and (1.14) for pure states. This is not in fact the case. We will prove a modest extension of the Balian-Low theorem which shows that there is no phase space localized projector E satisfying the three requirements Eqs.(1.17), (1.18) and (1.19). However, what we will do is find an “almost” complete set of phase space localized pure states from which we can construct a projector E that satisfies the exhaustivity condition Eq.(1.18) to a good approximation when acting on sufficiently decohered density operators. From this we can construct commuting operators \hat{X} and \hat{P} may with useful properties.

F. Earlier Work

Finally, we briefly mention a related approach. An earlier attempt to construct commuting position and momentum operators was considered in Ref.[21]. This construction involved doubling the original Hilbert space and then using operators defined on this enlarged space. (See also Ref.[22] for similar ideas). The resulting theory is essentially the same as ’t Hooft’s deterministic quantum theory [23]. However, this is no longer standard quantum theory. In the present work, by contrast, we stay within the framework of standard quantum theory.

G. This Paper

In Section 2, we briefly summarize some known properties of the Wigner function which help to make precise the idea that the state is sufficiently spread out in phase space. We also briefly note that the Wigner function naturally suggests an alternative method of defining commuting position and momentum operators. In Section 3, we prove a modest generalization of the Balian-Low theorem for projector operators. In Section 4 we introduce an orthonormal set of phase space localized states that are “almost” complete. We then use them to construct a set of phase space localized projection operators E_{nm} which are almost exhaustive. In Section 5 it is shown that the incompleteness does not matter if the density operator of the system is reasonably spread out in phase space. In Section 6, we use the projection operators E_{nm} to construct commuting position and momentum operators and compute the “distance” between these operators and the usual canonical pair, as in Eq.(1.9). In Section 7, we compare the probabilities for X and P with those for the usual position and momentum operators and find them to be close. We summarize and conclude in Section 8.

This contribution is based on the published work Ref.[24].

II. SOME PROPERTIES OF THE WIGNER FUNCTION

It will be useful for the rest of the paper to briefly summarize here some aspects of quantum mechanics in phase space and the Wigner function. Most of this is standard material and may be skipped by the informed reader, except the brief comments at the end of this section.

The Wigner representation for a density operator ρ (or indeed a wide class of operators) is defined by

$$W(p, q) = \frac{1}{2\pi\hbar} \int d\xi e^{-\frac{i}{\hbar}p\xi} \rho(q + \frac{1}{2}\xi, q - \frac{1}{2}\xi) \quad (2.1)$$

with inverse

$$\rho(x, y) = \int dp e^{\frac{i}{\hbar}p(x-y)} W(p, \frac{x+y}{2}) \quad (2.2)$$

Many calculations involving operators are usefully expressed in the Wigner representation. For example,

$$\text{Tr}(AB) = 2\pi\hbar \int dp dq W_A(p, q) W_B(p, q) \quad (2.3)$$

where $W_A(p, q)$ and $W_B(p, q)$ are the Wigner functions of A and B .

We will be interested in later sections in the behaviour of the density operator or Wigner function in a simple model of the decoherence process. We take the simplest case of a single particle coupled to a thermal environment in the limit of high temperature and negligible dissipation, with no external potential. The master equation for the density matrix $\rho(x, y)$ is,

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m} \left(\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2} \right) - \frac{D}{\hbar^2} (x - y)^2 \rho \quad (2.4)$$

where $D = 2m\gamma kT$. In the Wigner representation, the corresponding Wigner function obeys the equation,

$$\frac{\partial W}{\partial t} = -\frac{p}{m} \frac{\partial W}{\partial x} + D \frac{\partial^2 W}{\partial p^2} \quad (2.5)$$

Evolution according to the master equation Eq.(2.4) tends to produce approximate diagonality in position and momentum. In the Wigner representation, this appears as a spreading out phase space. Indeed, using Eq.(2.5) it may be shown that

$$(\Delta p)_t^2 = 2Dt + (\Delta p)_0^2 \quad (2.6)$$

$$(\Delta x)_t^2 = \frac{2}{3} \frac{Dt^3}{m^2} + (\Delta p)_0^2 \frac{t^2}{m^2} + \frac{2}{m} \sigma(x, p) + (\Delta x)_0^2 \quad (2.7)$$

where

$$\sigma(x, p) = \frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle \quad (2.8)$$

evaluated in the initial state. In particular, for long times, the phase space spreading behaves according to

$$\frac{(\Delta p)_t (\Delta x)_t}{\hbar} \sim \left(\frac{\gamma k T}{\hbar} \right) t^2 \quad (2.9)$$

This means that any initial state becomes spread out in phase space on the (typically very short) timescale $(\hbar/\gamma k T)^{1/2}$. (See Ref.[25] for similar calculations).

In the case above, a free particle, the spreading continues indefinitely. However, for a bound system with dissipation, equilibrium is eventually reached. For a simple harmonic oscillator at thermal equilibrium, for example, the ratio of thermal to quantum fluctuations is

$$\frac{(\Delta p)(\Delta x)}{\hbar} \approx \frac{k T}{\hbar \omega} \quad (2.10)$$

At typical laboratory temperatures and frequencies that are not unrealistically fast (for macroscopic systems), this number can be very large, of order 10^{10} say. These estimates will be relevant later on. In brief, they show that the density operator becomes very spread out in phase space (and hence slowly varying) very readily.

This much is known material and will be used later. However, we now note that the existence of the Wigner representation suggests an alternative method for constructing commuting position and momentum operators. It is well-known that when the Wigner function is sufficiently spread out, it becomes positive and may be regarded as a probability distribution for the variables p and q (which clearly commute, since they are numbers, not operators). The variables p and q do not correspond exactly to the operators \hat{p} and \hat{x} , but they are close when the Wigner function is spread out. In fact, it is easy to see that we have the following correspondences:

$$qW(p, q) \leftrightarrow \frac{1}{2} (\hat{x} \rho + \rho \hat{x}) \quad (2.11)$$

$$pW(p, q) \leftrightarrow \frac{1}{2} (\hat{p} \rho + \rho \hat{p}) \quad (2.12)$$

That is, multiplication by q or p in the Wigner representation corresponds to the operation of anticommutation with \hat{x} or \hat{p} on the density operator (and it is easy to show that these two operations on ρ commute). The point here is that these operators on ρ are operations on the space of density operators which have no counterpart in terms of operations on pure

states. This is therefore not a route to producing the desired pair of commuting operators envisaged by von Neumann.

III. AN EXTENSION OF THE BALIAN-LOW THEOREM

We first show that there is no projection operator satisfying the three properties Eqs.(1.18), (1.19) and (1.20). This is an almost trivial extension of the proof of the Balian-Low theorem given by Battle [17].

We consider the object $\text{Tr}(Exp)$, which we assume exists. (If it does not, i.e., is infinite, this means that E has infinite dispersion in either x or p , so is not phase space localized). We have, from the three properties of E_{mn} , Eqs.(1.17)-(1.19),

$$\begin{aligned}
\text{Tr}(Exp) &= \sum_{mn} \text{Tr}(ExE_{mn}p) \\
&= \sum_{mn} \text{Tr}(ExU_{mn}EU_{mn}^\dagger p) \\
&= \sum_{mn} \text{Tr}(EU_{mn}(x+na)E(p+bm)U_{mn}^\dagger) \\
&= \sum_{mn} \text{Tr}(E_{-m,-n}(x+na)E(p+bm)) \\
&= \sum_{mn} \text{Tr}(E_{-m,-n}xEp) \\
&= \text{Tr}(xEp) \\
&= \text{Tr}(Epx)
\end{aligned} \tag{3.1}$$

Or in other words,

$$\text{Tr}(E[x, p]) = 0 \tag{3.2}$$

which, via the commutation relations, implies that $\text{Tr}E = 0$. Since $E \geq 0$, this means that $E = 0$. So no projector satisfying the three properties exists, if one insists that they be satisfied exactly. Hence, as indicated we will relax the conditions in what follows.

We note in passing that this simple result has implications for the Balian-Low theorem in the pure state case. It may seem that one could avoid the Balian-Low theorem by dropping the requirement Eq.(1.10) and requiring that the states $|\psi_{nm}\rangle$ are generated from more than one fiducial state. However, one could then use those fiducial states (assuming they are orthogonal) to construct a projection operator E satisfying the properties Eqs.(1.17)-(1.19) and the above result shows that the Balian-Low theorem is not in fact avoided.

IV. AN ALMOST COMPLETE SET OF ORTHONORMAL PHASE SPACE LOCALIZED STATES

We now show how to construct an almost complete orthonormal set of phase space localized states, from which we can construct the projector E . Our starting point is the set of states considered by Low [16],

$$\psi_{nm}(x) = a^{-\frac{1}{2}} h(x - na) e^{2\pi i m x / a} \quad (4.1)$$

where $h(x)$ a window function on $[-\frac{1}{2}a, \frac{1}{2}a]$. These clearly satisfy the three conditions, Eqs.(1.10), (1.13) and (1.14). The Fourier transform of these wave function is

$$\tilde{\psi}_{nm}(p) = \left(\frac{2a}{\pi\hbar}\right)^{\frac{1}{2}} (-1)^m \frac{\sin(pa/2\hbar)}{(pa/\hbar - 2\pi m)} e^{-\frac{i}{\hbar} nap} \quad (4.2)$$

from which it is easy to see that $(\Delta p)^2$ diverges because $\tilde{\psi}_{nm}(p)$ goes to zero like $1/p$ for large p which is not fast enough. Differently put, the window function in Eq.(4.1) causes the derivative of the wave function to involve the δ -functions $\delta(x - na \pm \frac{1}{2}a)$, which are not square-integrable. These properties are fully in line with the Balian-Low theorem.

However, inspired by a suggestion of Zak [26], it is easy to see from Eq.(4.2) that we may take linear combinations of these states which fall off like $1/p^2$ and therefore have finite $(\Delta p)^2$. In particular, the states

$$\psi_{nm}^{(1)}(x) = \frac{1}{\sqrt{2}} (\psi_{n,2m}(x) + \psi_{n,2m+1}(x)) \quad (4.3)$$

have this property. One can also see this in configuration space: they vanish at $x = na \pm \frac{1}{2}a$, and the offending δ -function appearing in their derivative therefore causes no problems. They are not complete, consisting of just “half” the states in the momentum direction, and indeed the states missed out are the states

$$\chi_{nm}^{(1)}(x) = \frac{1}{\sqrt{2}} (\psi_{n,2m}(x) - \psi_{n,2m+1}(x)) \quad (4.4)$$

and these have infinite dispersion in p .

But it is not hard to see that the “remainder” states $\chi_{nm}^{(1)}$ decay like $1/p$ in momentum space, so a further process of “halving” is possible to produce more states with finite dispersion. That is, we define

$$\psi_{nm}^{(2)} = \frac{1}{\sqrt{2}} \left(\chi_{n,2m}^{(1)}(x) - \chi_{n,2m+1}^{(1)}(x) \right) \quad (4.5)$$

with new remainder states

$$\chi_{nm}^{(2)} = \frac{1}{\sqrt{2}} \left(\chi_{n,2m}^{(1)} + \chi_{n,2m+1}^{(1)} \right) \quad (4.6)$$

So the set of states $\psi_{nm}^{(1)}, \psi_{nm}^{(2)}, \chi_{nm}^{(2)}$ is complete and orthonormal, but the states $\psi_{nm}^{(1)}$ have infinite dispersion.

We can continue in this way to define the sequence of states

$$\psi_{nm}^{(K+1)} = \frac{1}{\sqrt{2}} \left(\chi_{n,2m+1}^{(K)} - \chi_{n,2m}^{(K)} \right) \quad (4.7)$$

and

$$\chi_{nm}^{(K+1)} = \frac{1}{\sqrt{2}} \left(\chi_{n,2m+1}^{(K)} + \chi_{n,2m}^{(K)} \right) \quad (4.8)$$

for $K = 1, 2, 3, \dots$. If we truncate the sequence at some finite value of K , $K = N$, say, then the set of states $\psi_{nm}^{(1)}, \psi_{nm}^{(2)} \dots \psi_{nm}^{(N)}$ together with the remainder states $\chi_{nm}^{(N)}$ are orthonormal and complete. We therefore have the completeness relation

$$\sum_{K=1}^N \sum_{n,m} \psi_{nm}^{(K)}(x) \psi_{nm}^{(K)}(y)^* + \sum_{n,m} \chi_{nm}^{(N)}(x) \chi_{nm}^{(N)}(y)^* = \delta(x - y) \quad (4.9)$$

In some sense, “most” of the states, namely the $\psi_{nm}^{(K)}$, have finite dispersion, and “some” of them, namely the $\chi_{nm}^{(N)}$ have infinite dispersion. In this way, as N increases, the infinite dispersion anticipated from the Balian-Low theorem is pushed into a progressively smaller set of states. (An interesting question is whether the limit $N \rightarrow \infty$ may be taken in any meaningful or useful way, but we will not pursue that here).

Since each state is a linear combination of the ψ_{nm} , one may also derive the following general formula,

$$\psi_{nm}^{(K)} = 2^{-K/2} \sum_{j=0}^{2^K-1} c_j^K \psi_{n,2^K m+j} \quad (4.10)$$

where

$$c_j^K = (-1)^j \quad \text{if } 0 \leq j \leq 2^{K-1} - 1 \quad (4.11)$$

and

$$c_j^K = -(-1)^j \quad \text{if } 2^{K-1} \leq j \leq 2^K - 1 \quad (4.12)$$

We also have

$$\chi_{nm}^{(K)} = 2^{-K/2} \sum_{j=0}^{2^K-1} (-1)^j \psi_{n,2^K m+j} \quad (4.13)$$

This may also be written,

$$\chi_{nm}^{(K)} = 2^{-K/2} \frac{(1 - e^{2^{K+1}\pi i x/a})}{(1 + e^{2\pi i x/a})} \psi_{n,2^{K+1}m} \quad (4.14)$$

The states $|\psi_{nm}^{(K)}\rangle$ are not all obtained from a single fiducial state, since we have

$$|\psi_{nm}^{(K)}\rangle = U_{nm}^{(K)} |\psi^K\rangle \quad (4.15)$$

where

$$U_{nm}^{(K)} = U_{n,2^K m} \quad (4.16)$$

and

$$|\psi^K\rangle = |\psi_{00}^K\rangle \quad (4.17)$$

There are therefore N fiducial states for the set $\psi_{nm}^{(K)}$, $K = 1, \dots, N$.

At some length, one can compute the averages and dispersions of p and x in the fiducial states $|\psi^K\rangle$. One obtains

$$\langle p \rangle_K = \frac{2\pi\hbar}{a} \left(2^{K-1} - \frac{1}{2} \right) \quad (4.18)$$

$$\langle x \rangle_K = 0 \quad (4.19)$$

$$(\Delta p)_K^2 = \left(\frac{2\pi\hbar}{a} \right)^2 \frac{(2^{2K} - 1)}{12} \quad (4.20)$$

$$(\Delta x)_K^2 = \frac{a^2}{12} \quad (4.21)$$

(Note that $\langle p \rangle_K$ is not zero. Because there is more than one fiducial state, it does not appear to be possible to shift the states so that $\langle p \rangle_K = 0$ in the fiducial states without spoiling orthogonality.)

The construction described above is concisely summarized as follows: For each set of 2^N lattice points in the momentum direction, there are $2^N - 1$ states with finite dispersion and just 1 state (the remainder state), with infinite dispersion.

Consider the following simple example to illustrate the construction. Consider the 8 lattice points $m = 0, 1 \dots 7$ in the momentum direction, so $N = 3$. Then the 8 states which

depend only on these points are

$$\begin{aligned}
\psi_{n,0}^{(1)} &= \frac{1}{\sqrt{2}} (\psi_{n,0} + \psi_{n,1}) \\
\psi_{n,1}^{(1)} &= \frac{1}{\sqrt{2}} (\psi_{n,2} + \psi_{n,3}) \\
\psi_{n,2}^{(1)} &= \frac{1}{\sqrt{2}} (\psi_{n,4} + \psi_{n,5}) \\
\psi_{n,3}^{(1)} &= \frac{1}{\sqrt{2}} (\psi_{n,6} + \psi_{n,7})
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\psi_{n,0}^{(2)} &= \frac{1}{2} (\psi_{n,0} - \psi_{n,1} - \psi_{n,2} + \psi_{n,3}) \\
\psi_{n,1}^{(2)} &= \frac{1}{2} (\psi_{n,4} - \psi_{n,5} - \psi_{n,6} + \psi_{n,7})
\end{aligned} \tag{4.23}$$

$$\psi_{n,0}^{(3)} = \frac{1}{2^{3/2}} (\psi_{n,0} - \psi_{n,1} + \psi_{n,2} - \psi_{n,3} - \psi_{n,4} + \psi_{n,5} - \psi_{n,6} + \psi_{n,7}) \tag{4.24}$$

$$\chi_{n,0}^{(3)} = \frac{1}{2^{3/2}} (\psi_{n,0} - \psi_{n,1} + \psi_{n,2} - \psi_{n,3} + \psi_{n,4} - \psi_{n,5} + \psi_{n,6} - \psi_{n,7}) \tag{4.25}$$

There are 7 finite dispersion states and one infinite dispersion state. It is easy to see that they are orthonormal and form a complete set on these 8 lattice points. Note also that widths of the states $\psi_{nm}^{(K)}$ increases with K , but this is in some sense offset by the fact that the states become progressively sparser.

So far we have been working with a complete set of states. We now come to the specific form of the proposal to relax the requirement of using a complete set of states (in the construction of commuting operators \hat{P} and \hat{X}): we quite simply drop the remainder states $\chi_{nm}^{(N)}$, which have infinite dispersion, and use only the incomplete set $\psi_{nm}^{(K)}$, $K = 1, 2 \dots N$, which have finite dispersion. That is, in each set of 2^N lattice points in the momentum direction, we use $2^N - 1$ out of the 2^N states.

It seems likely that this approximation will be valid for sufficiently large N , for suitable density matrices. We will show in the next section that reasonable results can be obtained using the $\psi_{nm}^{(K)}$ alone. Here, we briefly look at the remainder states $\chi_{nm}^{(N)}$ to see under what conditions they may be dropped. Note first that the states $\psi_{nm}^{(K)}(x)$ have the property that they vanish at the end-points of the intervals, $x = na \pm a/2$ (as they must, so that their derivative does not have a δ -function). Not surprisingly therefore, the remainder states $\chi_{nm}^{(N)}(x)$ become narrower and progressively more concentrated about the end-points as $N \rightarrow \infty$, as one can see from Eq.(4.14) (since they are in some sense making up for the

fact that the $\psi_{nm}^{(K)}(x)$ vanish at the end-points, and the whole set of states is complete). This suggests that, under suitably coarse grained conditions, the behaviour of these states at single points will become insignificant. We will see this more explicitly below.

We may now use the almost complete set of states to construct the desired projection operators E_{nm} localized on phase space cells. We choose the phase space cells to have 2^N lattice points in the p -direction and 1 lattice point in the x -direction. We have found $2^N - 1$ states with finite dispersion in each of those cells. We therefore take the projector E to be

$$E = \sum_{K=1}^N \sum_{m=0}^{2^{N-K}-1} |\psi_{0m}^{(K)}\rangle \langle \psi_{0m}^{(K)}| \quad (4.26)$$

It is a sum over all states depending only on the lattice points 0 to $2^N - 1$ (in the momentum direction). Importantly, this is an *exact* projection operator which localizes onto a region of phase space – it satisfies

$$E^2 = E \quad (4.27)$$

exactly. The projection operator for any other cell is easily obtained by unitary displacement, using steps of size 2^N lattice points in the momentum direction (and single steps in the x -direction):

$$E_{nm} = U_{nm}^{(N)} E (U_{nm}^{(N)})^\dagger \quad (4.28)$$

These projectors clearly satisfy the exclusivity condition, Eq.(1.17), exactly, but do not satisfy the exhaustivity condition Eq.(1.18) since we have

$$\sum_{nm} E_{nm} + \sum_{n,m} |\chi_{nm}^{(N)}\rangle \langle \chi_{nm}^{(N)}| = 1 \quad (4.29)$$

But, as we have argued, we expect the χ terms to be negligible under suitable conditions so the projectors E_{nm} should be almost exhaustive

$$\sum_{nm} E_{nm} \approx 1 \quad (4.30)$$

The approximate nature of this property may not in fact matter for many practical applications. The phase space projector onto a large cell Γ in phase space is defined by

$$E_\Gamma = \sum_{n,m \in \Gamma} E_{nm} \quad (4.31)$$

The projector onto the region outside Γ is then defined to be

$$\bar{E}_\Gamma = 1 - E_\Gamma \quad (4.32)$$

so they are trivially exhaustive, $E_\Gamma + \bar{E}_\Gamma = 1$. The key point here is that the remainder states dropped in the construction of E have infinite dispersion so they do not naturally belong in the construction of a phase space projector for a finite region of phase space.

We will need some further properties of E . We have

$$\text{Tr} E = 2^N - 1 \quad (4.33)$$

which means it projects onto a phase space region of size $(2^N - 1)(2\pi\hbar)$. The object $E/\text{Tr} E$ may be thought of as a density operator and we can compute averages and variances to see the properties of E . We have

$$\langle \hat{p} \rangle_E = \frac{\text{Tr}(pE)}{\text{Tr} E} \quad (4.34)$$

$$= \frac{2\pi\hbar}{a} \left(2^{N-1} - \frac{1}{2} \right) \quad (4.35)$$

and

$$(\Delta p)_E^2 \approx \sum_{K=1}^N \frac{(\Delta p)_K^2}{2^K} \quad (4.36)$$

$$\approx \frac{2^{N+1}\pi^2\hbar^2}{3a^2} \quad (4.37)$$

where we show only the leading terms of large N . We also have

$$\langle \hat{x} \rangle_E = 0 \quad (4.38)$$

$$(\Delta x)_E^2 = \frac{a^2}{12} \quad (4.39)$$

Since $\langle \hat{p} \rangle_E \neq 0$, it is in fact useful to perform a simple translation in momentum and define a related projector E' with all the same properties as E except that $\text{Tr}(pE') = 0$. We will use this in what follows.

The construction of an exact projector with the above properties is the main achievement of this section and will be used to construct the commuting \hat{X} and \hat{P} operators below.

Finally, we make two minor remarks. First, note that position and momentum enter in the construction in very different ways. However, the constant a is arbitrary, so may be tuned to make the width of the projector E arbitrarily small in either the x or p direction. Also, the states Eq.(4.1) and (4.2) are a Fourier transform pair, so we could easily interchange them and start with a set of states that have perfect localization in p , instead of x . It would of course be of interest to find a construction in which x and p entered on an equal footing.

Second, we note for comparison that Omnès has made extensive use of phase quasi-projectors of the form

$$P_\Gamma = \int_\Gamma dp dq |p, q\rangle \langle p, q| \quad (4.40)$$

where $|p, q\rangle$ are phase space localized states, such as the coherent states [9]. These have proved very useful for discussing emergent classicality in quantum theory. However, in contrast to the projectors constructed here, these are not exact projectors, since they obey the approximate relation

$$P_\Gamma^2 \approx P_\Gamma \quad (4.41)$$

(and so P_Γ and $1 - P_\Gamma$ are only approximately exclusive). It would be of interest to revisit some of Omnès results using the exact projectors constructed here.

V. VALIDITY OF APPROXIMATE COMPLETENESS

Now we come to a crucial check of our approach, which is to determine the conditions under which working with an approximately complete set of states gives reasonable results. The density matrix ρ of the system satisfies $\text{Tr} \rho = 1$. If the states $|\psi_{nm}^{(K)}\rangle$ for $K = 1, \dots, N$ are approximately complete, then we should have

$$\sum_{K=1}^N \sum_{nm} \langle \psi_{nm}^{(K)} | \rho | \psi_{nm}^{(K)} \rangle \approx 1 \quad (5.1)$$

which is the same as

$$\sum_{nm} \text{Tr} (E_{nm} \rho) \approx 0 \quad (5.2)$$

We check this. It is most useful to exploit the Wigner representation, Eqs.(2.1),(2.2) together with the property Eq.(2.3), so we have

$$\sum_{K=1}^N \sum_{nm} \langle \psi_{nm}^{(K)} | \rho | \psi_{nm}^{(K)} \rangle = 2\pi\hbar \sum_{K=1}^N \sum_{nm} \int dp dq W^{(K)}(p, q) W_\rho(p + \frac{2^{K+1}\pi\hbar}{a}m, q + na) \quad (5.3)$$

where W_ρ is the Wigner function of ρ and $W^{(K)}$ is the Wigner function of $|\psi^K\rangle$. Now the crucial step is to approximate the discrete sum over n, m with an integral over continuous variables, $\bar{p} = 2^{K+1}\pi\hbar m/a$, $\bar{q} = na$ and we have

$$\sum_{nm} W_\rho(p + \frac{2^{K+1}\pi\hbar}{a}m, q + na) \approx \int \frac{d\bar{p}d\bar{q}}{2^{K+1}\pi\hbar} W_\rho(p + \bar{p}, q + \bar{q}) \quad (5.4)$$

$$= \frac{1}{2^{K+1}\pi\hbar} \quad (5.5)$$

This approximation is valid as long as the Wigner function of ρ is slowly varying over phase space volumes of size $2^K(2\pi\hbar)$. Since K runs up to N , we require slow variation on a scale of size $2^N(2\pi\hbar)$. This is typically the case for density matrices that are sufficiently decohered, as we saw in Section 2. We now have

$$\sum_{K=1}^N \sum_{nm} \langle \psi_{nm}^{(K)} | \rho | \psi_{nm}^{(K)} \rangle \approx \sum_{K=1}^N \frac{1}{2^K} \quad (5.6)$$

$$= 1 - \frac{1}{2^N} \quad (5.7)$$

Hence we do indeed get a result close to 1 as long as N is sufficiently large.

One other related result is worth recording here since it will be used in the next section. Any density operator satisfies the relation

$$\int \frac{dkdq}{2\pi\hbar} U^\dagger(q, k) \rho U(q, k) = 1 \quad (5.8)$$

where $U(q, k)$ is the unitary shift operator in phase space. This is easily proved using the Wigner representation above, since we have

$$\langle x | U^\dagger(q, k) \rho U(q, k) | y \rangle = \int dp e^{\frac{i}{\hbar} p(x-y)} W_\rho(p - k, \frac{x+y}{2} - q) \quad (5.9)$$

and integrating the right-hand side over k and q yields $2\pi\hbar\delta(x-y)$.

One would expect that, for sufficiently slowly varying Wigner functions, a discrete version of the result Eq.(5.8) would hold. So suppose instead of the operator $U(q, k)$, we use the operators $U_{nm}^{(K)}$ defined in Eq.(4.16). We then have

$$\langle x | (U_{nm}^{(K)})^\dagger \rho U_{nm}^{(K)} | y \rangle = \int dp e^{\frac{i}{\hbar} p(x-y)} W_\rho(p - \frac{2^{K+1}\pi\hbar}{a}m, \frac{x+y}{2} - na) \quad (5.10)$$

Using the same approximation and same steps as in Eq.(5.4) to do the sum over n, m , it follows that

$$\sum_{nm} (U_{nm}^{(K)})^\dagger \rho U_{nm}^{(K)} \approx \frac{1}{2^K} \quad (5.11)$$

and therefore

$$\sum_{K=1}^N \sum_{nm} (U_{nm}^{(K)})^\dagger \rho U_{nm}^{(K)} \approx 1 - \frac{1}{2^N} \quad (5.12)$$

VI. CONSTRUCTION OF COMMUTING POSITION AND MOMENTUM OPERATORS

We now use the projection operator E'_{nm} to construct commuting position and momentum operators, as outlined in the introduction. They are,

$$\hat{X} = \sum_{nm} X_n E'_{nm} \quad (6.1)$$

$$\hat{P} = \sum_{nm} P_m E'_{nm} \quad (6.2)$$

where $X_n = na$ and $P_m = m \times 2^N (2\pi\hbar/a)$. Clearly

$$[\hat{X}, \hat{P}] = 0 \quad (6.3)$$

as required.

We need to determine whether these operators are close to the original canonical pair, \hat{x}, \hat{p} . To do this we need some measure of distance, $\|\hat{P} - \hat{p}\|$. We will define this by

$$\|\hat{P} - \hat{p}\|^2 = \text{Tr} \left((\hat{P} - \hat{p})^2 \rho \right) \quad (6.4)$$

(and similarly for \hat{X}) where ρ is the density operator of the system, which we will assume is reasonably spread out in phase space. A more general measure of distance would include some sort of optimization over ρ , but, for the class of reasonably decohered density operators, the result depends only weakly on ρ .

We have

$$\|\hat{P} - \hat{p}\|^2 = \text{Tr} \left((\hat{P}^2 - 2\hat{P}\hat{p} + \hat{p}^2) \rho \right) + \text{Tr} \left(\hat{P}[\hat{p}, \rho] \right) \quad (6.5)$$

where the operators \hat{P} are moved to the left in the first term in preparation for inserting the explicit form for \hat{P} below (since $[\hat{P}, \hat{p}] \neq 0$). It is useful to introduce the notation $\text{tr}(\dots)$ to denote a trace over the incomplete set of states $|\psi_{nm}^{(K)}\rangle$. We may then write

$$\|\hat{P} - \hat{p}\|^2 = \text{tr} \left((\hat{P}^2 - 2\hat{P}\hat{p} + \hat{p}^2) \rho \right) + d_p^2 + \text{Tr} \left(\hat{P}[\hat{p}, \rho] \right) \quad (6.6)$$

where

$$d_p^2 = \text{Tr}(\hat{p}^2 \rho) - \text{tr}(\hat{p}^2 \rho) \quad (6.7)$$

In the previous section we saw that $\text{Tr} \rho$ and $\text{tr} \rho$ are both very close (to order 2^{-N}) and it is reasonable to expect the same if ρ is replaced with $\hat{p}^2 \rho$, so d_p^2 will be negligible and we drop it.

We also expect the term $\text{Tr}(\hat{P}[\hat{p}, \rho])$ to be small, since a decohered ρ will be approximately diagonal in momentum. Inserting the explicit expression for \hat{P} , it is easily shown that

$$\text{Tr}(\hat{P}[\hat{p}, \rho]) = \sum_{nm} P_m \text{Tr}([\hat{p}, (U_{nm}^{(N)})^\dagger \rho U_{nm}^{(N)}]) \quad (6.8)$$

Since we are working in the approximation in which ρ is sufficiently slowly varying that the sum over n becomes an integral. It is then easily seen (using Eq.(5.8, for example) that, in this approximation, the object

$$\sum_n (U_{nm}^{(N)})^\dagger \rho U_{nm}^{(N)} \quad (6.9)$$

is diagonal in momentum. Therefore the term Eq.(6.8) is zero to the approximation we are using.

Consider the one remaining term in Eq.(6.6). Inserting the explicit for for \hat{P} , we have

$$\|\hat{P} - \hat{p}\|^2 = \sum_{nm} \text{tr}(E'_{nm} (P_m - \hat{p})^2 \rho) \quad (6.10)$$

$$= \sum_{nm} \text{tr}(E' \hat{p}^2 (U_{nm}^{(N)})^\dagger \rho U_{nm}^{(N)}) \quad (6.11)$$

From Eq.(5.11), we have

$$\sum_{nm} (U_{nm}^{(N)})^\dagger \rho U_{nm}^{(N)} \approx \frac{1}{2^N} \quad (6.12)$$

and so

$$\|\hat{P} - \hat{p}\|^2 \approx (\Delta p)_E^2 \quad (6.13)$$

$$\approx \frac{2^{N+1} \pi^2 \hbar^2}{3a^2} \quad (6.14)$$

where only the leading term for large N is given.

A similar (and simpler) calculation shows that

$$\|\hat{X} - \hat{x}\|^2 \approx (\Delta x)_E^2 \quad (6.15)$$

$$= \frac{a^2}{12} \quad (6.16)$$

Putting all these results together we obtain

$$\|\hat{P} - \hat{p}\| \cdot \|\hat{X} - \hat{x}\| \approx C \hbar \quad (6.17)$$

where

$$C = 2^{N/2} \frac{\pi}{3\sqrt{2}} \quad (6.18)$$

This result is valid in the approximation of large N and for slowly varying density operators.

Eqs. (5.7) and (6.17) show that N needs to be chosen to be “large” to get approximate completeness, yet “small” for the commuting operators to be close to the original canonical pair. It seems likely, however, that there is a range of intermediate values that will meet both of these requirements. For example, take $N = 20$. Then $C \sim 10^3$, safely within the limit estimated in Eq.(1.3), so the difference between the old and new operators will be completely invisible to macroscopic observations. The error due to approximate completeness is of order 2^{-N} , which is about 10^{-6} . So there appears to be a large regime in which the approach works well.

VII. PROBABILITIES FOR POSITION AND MOMENTUM

The results Eq.(5.7) and Eq.(6.17) are only indicative, and the true test of closeness of the old and new operators is a comparison of the probabilities for x, p and X, P .

Consider the probability that the variable X lies in the range Δ_X , where Δ_X is the interval $[n_1 a, n_2 a]$, for some integers n_1, n_2 . The probability is

$$p(\Delta_X) = \text{Tr}(\rho E_{\Delta_X}) \quad (7.1)$$

where the projector E_{Δ_X} is defined by

$$E_{\Delta_X} = \sum_{n \in \Delta_X} \sum_m E'_{nm} \quad (7.2)$$

(and the loose notation $n \in \Delta_X$ means $n \in [n_1, n_2]$). The probability for lying outside the region Δ_X is defined using the projector $1 - E_{\Delta_X}$, so that the probabilities add to 1 exactly, and the approximate completeness discussed earlier poses no problems. The probability for P is similarly defined, in terms of a projector E_{Δ_P} defined by

$$E_{\Delta_P} = \sum_{m \in \Delta_P} \sum_n E'_{nm} \quad (7.3)$$

Both of these probabilities therefore involve the probability p_{nm} associated with our basic phase space cell of size $2^N(2\pi\hbar)$, given by

$$p_{nm} = \text{Tr}(E'_{nm}\rho) = \text{Tr}\left(E' (U_{nm}^{(N)})^\dagger \rho U_{nm}^{(N)}\right) \quad (7.4)$$

It is more usefully written in the Wigner representation,

$$p_{mn} = 2\pi\hbar \int dp dq W_{E'}(p, q) W_{\rho}(p + P_m, q + X_n) \quad (7.5)$$

where, recall, $X_n = na$ and $P_m = m \times 2^N(2\pi\hbar/a)$.

The probability for the variable X is

$$p(\Delta_X) = \sum_{n \in \Delta_X} \sum_m p_{nm} \quad (7.6)$$

Since the Wigner function of ρ is assumed slowly varying, the sum over m may be approximated by an integral over P_m regarded as a continuous variable. The p in $p + P_m$ is absorbed into the integration and we obtain,

$$p(\Delta_X) \approx \frac{a}{2^N} \sum_{n \in \Delta_X} \int dp dq W_{E'}(p, q) \rho(q + X_n, q + X_n) \quad (7.7)$$

Now the p integral may be carried out with the result

$$p(\Delta_X) \approx a \sum_{n \in \Delta_X} \int dq \frac{\langle q | E' | q \rangle}{2^N} \rho(q + X_n, q + X_n) \quad (7.8)$$

Since E' is phase space localized, one can see that the first part of the integrand is a smearing function peaked about $q = 0$ and with width $(\Delta q)^2 \approx a^2/12$. It is normalized to 1 (for large N) when integrated over q since $\text{Tr} E' = 2^N - 1$. If we assume that

$$\Delta_X^2 \gg \frac{a^2}{12} \quad (7.9)$$

then the presence of the smearing function makes no difference and we obtain

$$p(\Delta_X) \approx a \sum_{n \in \Delta_X} \rho(na, na) \quad (7.10)$$

If, as we assume, the density operator is sufficiently slowly varying for the discrete sum to become an integral, we obtain

$$p(\Delta_X) \approx \int_{\Delta_X} dX \rho(X, X) \quad (7.11)$$

the usual probability for position.

Similarly for P , with analogous approximations, we obtain

$$p(\Delta_P) = 2^N \left(\frac{2\pi\hbar}{a} \right) \sum_{m \in \Delta_P} \int dp \frac{\langle p | E' | p \rangle}{2^N} \tilde{\rho}(p + P_m, p + P_m)$$

$$\approx 2^N \left(\frac{2\pi\hbar}{a} \right) \sum_{m \in \Delta_P} \tilde{\rho}(P_m, P_m) \quad (7.12)$$

where $\tilde{\rho}(p, p')$ is the Fourier transform of the density matrix $\rho(x, y)$. Again, for slowly varying density operators the discrete sum becomes an integral and we obtain

$$p(\Delta_P) \approx \int_{\Delta_P} dP \tilde{\rho}(P, P) \quad (7.13)$$

which coincides with the usual probability for p .

We therefore find that the probabilities for X, P coincide with those for x, p as long as the following conditions hold:

- (i) N is sufficiently large that the errors $1/2^N$ are tolerably small
- (ii) The density operator ρ is slowly varying on scales of size $2^N(2\pi\hbar)$.
- (iii) The widths of the projections satisfy $\Delta_X \Delta_P \gg 2^N(2\pi\hbar)$.

The key restriction is (ii), the restriction on the density operator. Eqs.(2.9) and (2.10) indicate that the density operator can easily become sufficiently broad for (ii) to be satisfied. For example, for a time-evolving state, (ii) will be satisfied for

$$t \gg \left(\frac{\hbar}{\gamma kT} \right)^{1/2} 2^{N/2} \quad (7.14)$$

This can easily be extremely short, even for large N . Similarly, for a state close to thermal equilibrium, (ii) is satisfied for

$$\frac{kT}{\hbar\omega} \gg 2^N \quad (7.15)$$

which is again easily satisfied.

VIII. SUMMARY AND DISCUSSION

We have constructed a pair of commuting operators \hat{X}, \hat{P} which, at sufficiently coarse grained scales, are close (in a variety of ways) to the canonical position and momentum operators \hat{x}, \hat{p} . These commuting operators offer a new way of defining the relationship between approximate and exact decoherence.

There are two ways in which this programme could be developed and improved. First of all, this paper has concentrated on the technicalities of constructing \hat{X} and \hat{P} . It would be

useful to develop more details of the conceptual framework in which they are used to discuss emergent classicality.

Secondly, the present approach works at the coarse grained scales of about 2^N phase space cells. Although arguably small to classical eyes, there are some ways in which this scale is quite large, and there are indications that some version of the present approach should work at finer scales. For example, it is known that the Wigner function (and also the P -function) become positive when coarse grained over just one or two phase space cells (rather than 2^N cells, as here), a feature that is often taken as indicator of approximate decoherence and emergent classicality [27]. This suggests that there might be another way to construct commuting position and momentum operators which does not require such a large amount of phase space coarse graining. It would, for example, be of interest to see if von Neumann's original suggestion (involving an explicit orthogonalization of the coherent states) can actually be made to work.

These and related questions will be pursued in future publications.

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- [1] For general discussions of the emergence of classical behaviour from quantum theory, see Halliwell JJ (2005) *Contemp.Phys.* **46** 93, and Hartle J B (1994) in *Proceedings of the Cornelius Lanczos International Centenary Conference*, edited by J.D.Brown, M.T.Chu, D.C.Ellison and R.J.Plemmons (SIAM, Philadelphia).
 - [2] Von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton).
 - [3] Joos E and Zeh H D 1985 *Z.Phys.* **B59** 223.
 - [4] Halliwell J J and Zoupas A 1995 *Phys.Rev.* **D52** 7294; 1997 *Phys.Rev.* **D55** 4697.

- [5] Gell-Mann M and Hartle J B 1990, in *Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity*, Vol. VIII, edited by Zurek, W (Addison Wesley, Reading, MA).
- [6] Gell-Mann M and Hartle J B 1993 *Phys.Rev.* **D47** 3345.
- [7] Griffiths R B 1984, *J.Stat.Phys.* **36** 219; 1993 *Phys.Rev.Lett.* **70** 2201; 1996 *Phys.Rev.* **A54** 2759; 1998 **A57** 1604.
- [8] Omnès R 1992 *Rev.Mod.Phys.* **64** 339 and references therein.
- [9] Omnès R 1997 *J. Math. Phys.* **38** 697.
- [10] Halliwell J J 1996, in *Fundamental Problems in Quantum Theory*, edited by Greenberger D and Zeilinger A, *Annals of the New York Academy of Sciences* **775** 726.
- [11] Dowker H F and Kent A 1996 *J.Stat.Phys.* **82** 1575; 1995 *Phys.Rev.Lett.* **75** 3038.
- [12] Perelomov A M 1971 *Teor.Mat.Fiz* **6** 213.
- [13] Bargmann V, Butera P, Giradello L and Klauder J R 1971 *Rep.Math.Phys.* **2** 221.
- [14] Bacry H, Grossmann A and Zak J 1975 *Phys.Rev.* **B12** 1118.
- [15] Balian R 1981 *C.R.Acad.Sci. (Paris)* **292** 1357.
- [16] Low F 1985, in *Passion for Physics – Essays in Honor of Geoffrey Chew*, edited by DeTar C, Finkelstein J and Chung-I Tan (World Scientific, Singapore).
- [17] Battle G 1988 *Lett.Math.Phys.* **15** 175.
- [18] Zak J 2001 *J.Phys.* **A34** 1063.
- [19] Zak J 2003 *J.Phys.* **A36** L553.
- [20] Zak J 1967 *Phys.Rev.Lett.* **19** 1385.
- [21] Halliwell J J 2001 *Phys.Rev.* **D63** 085013.
- [22] Haagersup U and Rordam M 1995 *Duke Math.J.* **77** 627.
- [23] 't Hooft G 1999 *Class.Quant.Grav.* **16** 3263.
- [24] Halliwell J J 2005 *Phys.Rev.* **A72** 042109.
- [25] Anastopoulos C and Halliwell J J 1995 *Phys.Rev.* **D51** 6870.
- [26] Zak J, private communication.
- [27] Diósi L and Kiefer C 2002 *J.Math.Phys.* **A35** 2675.